1.6: Substitution Method and Exact Equations

As is usual with any substitution method, we wish to substitute in a new variable $v = \alpha(x, y)$ into the differential equation $dy/dx = f(x, y)$ so that the new differential equation $dv/dx = g(x, y)$ is one that we know how to solve.

Example 1. Using substitution, solve the differential equation

$$\frac{dy}{dx} = (x + y + 3)^2.$$

Definition 1. A **homogeneous** first-order differential equation is one that can be written in the form $\frac{dy}{dx} = f(\frac{y}{x})$. In this case we use the substitution $v = \frac{y}{x}$ so that

$$y = vx, \quad \text{and} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Example 2. Solve the differential equation

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2. \tag{1}$$

Exercise 1. Solve the initial value problem

$$x\frac{dy}{dx} = y + \sqrt{x^2 - y^2}, \quad y(x_0) = 0 \quad (x_0 > 0).$$

Definition 2. A first-order differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{2}$$

is called a **Bernoulli** equation. Using the substitution $v = y^{1-n}$ (2) becomes

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

which we can solve using the methods from the previous section.

Example 3. Rewriting the (1) in the form

$$\frac{dy}{dx} - \frac{3}{2x}y = \frac{2x}{y}$$

solve using the method described above.

Exercise 2. Use the method of Bernoulli equations to solve

$$x\frac{dy}{dx} + 6y = 3xy^{4/3}.$$

Example 4.

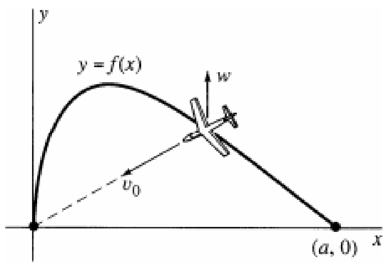

FIGURE 1.6.4. The airplane headed for the origin.

An airplane departs from point $(a, 0)$ located due East of its intended destination at the origin. The wind is blowing due North with a constant speed of w. We assume the plane is alway pointed at its destination so that its velocity vector v_0 is also. (Figure 1.6.4)

Hence the trajectory $y = f(x)$ of the plane satisfies

$$\frac{dy}{dx} = \frac{1}{v_0 x}\left(v_0 y - w\sqrt{x^2 + y^2}\right). \qquad (3)$$

Let $k = \frac{w}{v_0}$ and the substitution $y = xv$ to show that (3) has the solution

$$y(x) = \frac{a}{2}\left[\left(\frac{x}{a}\right)^{1-k} - \left(\frac{x}{a}\right)^{1+k}\right]. \qquad (4)$$

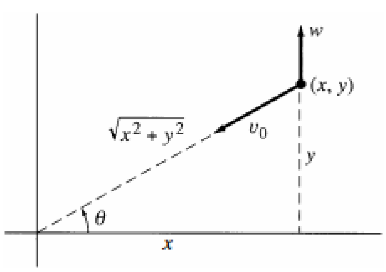

FIGURE 1.6.5. The components of the velocity vector of the airplane.

Exercise 3. Using (4), find the maximum amount by which the plane is blown of course when a=200 mi, v_0=500 mi/h, and w= 100 mi/h.

Definition 3. An **exact** differential equation is of the form

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0 \quad \text{or} \quad M(x,y)dx + N(x,y)dy = 0, \tag{5}$$

where $F(x,y)$ is a differentiable function of x and y. Recall that if F is twice differentiable then a necessary condition is that

$$\frac{\partial M}{\partial y} = F_{xy} = F_{yx} = \frac{\partial N}{\partial x}.$$

Example 5. Find the general solution to the differential equation

$$y^3 dx + 3xy^2 dy = 0.$$

Theorem 1. (Criterion for Exactness) Suppose that the functions $M(x,y)$ and $N(x,y)$ are continuous and have continuous partial derivatives on the open rectangle R. Then the differential equation (5) is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Homework. Example 9 from the book. Read the Section and Examples on Reducible Second-Order Equations. 1-21, 31-39, 43-51, 57-61 (odd)